UNIVERSITY OF WAIKATO

Hamilton
New Zealand

A New Framework for Yield Curve,

Output and Inflation Relationships

Leo Krippner

Department of Economics

Working Paper in Economics 7/05

December 2005

Leo Krippner
AMP Capital Investors
and University of Waikato

Email: leo.krippner@ampcapital.com

Abstract

This paper develops a theoretically-consistent and easy-to-apply framework for interpreting, investigating, and monitoring the relationships between the yield curve, output, and inflation. The framework predicts that steady-state inflation plus steady-state output growth should be cointegrated with the long-maturity level of the yield curve as estimated by a arbitrage-free version of the Nelson and Siegel (1987) model, while the shape of the yield curve model from that model should correspond to the profile (that is, the timing and magnitude) of expected future inflation and output growth. These predicted relationships are confirmed empirically using 51 years of United States data. The framework may be used for monitoring expectations of inflation and output growth implied by the yield curve. It should also provide a basis for using the yield curve to value and hedge derivatives on macroeconomic data.

Keywords

yield curve
term structure of interest rates
inflation
real output growth
Nelson and Siegel model
Heath-Jarrow-Morton framework

JEL Classification

E31, E32, E43

1 Introduction

This article develops a macroeconomic-finance framework based on a standard continuous-time economy and an arbitrage-free version of the popular and parsimonious Nelson and Siegel (1987) (hereafter NS) model of the yield curve. The motivation is to provide a theoretically-consistent, yet easy-to-apply foundation for interpreting, investigating, and monitoring the relationships between the yield curve, output, and inflation.

By context, yield curve, output, and inflation relationships have already been well-established in the existing literature, but often as a statistical exercise with limited theoretical structure. That is, single-equation ordinary least squares (OLS) regressions typically show a strong relationship between the current slope of the yield curve (often measured as the 10-year less the 90-day interest rate) and future output growth, a moderate relationship between the current slope of the yield curve and future inflation, and a cointegrating relationship between term interest rates and inflation.[1] However, as noted in Estrella (2003) pp. 1-4, the various justifications advanced for these empirical relationships are generally informal or heuristic: e.g real business cycles, counter-cyclical monetary policy, and life-cycle consumption to justify yield curve/output relationships; and the Fisher hypothesis with assumed constant or stationary real interest rates to justify yield curve/inflation relationships and interest rate/inflation cointegration. More formal macroeconomic foundations include Rendu de Lint and Stolin (2003) and Estrella (2003),[2] but the yield curves in those models are not constructed to be arbitrage-free.

An alternative approach to investigating yield curve, output, and inflation relationships is to use vector autoregressive (VAR) models containing selected macroeconomic and yield curve data, such as in Bernard and Gerlach (1998), Ang and Piazzesi (2003), Jardet (2004), and Diebold, Rudebusch and Aruoba (2005). VAR models allow some theoretical structure to be imposed via parameter restrictions, including an arbitrage-free construction in the Ang and Piazzesi (2003) model. That said, the models noted still have an implicit atheoretical element given that VAR dynamics are assumed, as is the order of the autoregression and the parameter restrictions, rather than being derived from an underlying theoretical structure. Also, a practical issue associated with VAR models is that they can be challenging to estimate and interpret due to their lack of parsimony, even after simplifying restrictions are imposed to prevent overfitting and avoid parameter instability.[3]

Conversely, the framework developed in this article allows parsimonious single-equation econometric relationships between the yield curve, output, and inflation to be explicitly derived. This embeds the theoretical consistency of the underlying continuous-time models of the economy and the yield curve, and also provides theoretical parameter values to compare against the empirically estimated values. The estimation process itself simply requires "fitting" the yield curve data at each point in time (an approach familiar to users of NS models),[4] and then using the resulting output to estimate (via OLS) the derived econometric relationships with output and inflation data.

The article proceeds as follows: section 2 develops a generic multifactor version of the standard continuous-time general-equilibrium-economy model of the yield curve from Cox, Ingersoll and Ross

---

[1]Berk (1998) provides a useful survey. Recent examples reporting these results include: Hamilton and Kim (2002), Bordo and Haubrich (2004), Nakaota (2005) and Paya, Matthews and Peel (2005) for yield curve/output; Estrella, Rodrigues and Schich (2003) for yield curve/inflation; and Fahmy and Kandi (2003) and Lai (2004) for interest rate/inflation cointegration.

[2]Rendu de Lint and Stolin (2003) derives a yield curve/output relationship via an intertemporal production and endowment economy, and Estrella (2003) derives yield curve, output, and inflation relationships via a standard dynamic model of the macroeconomy incorporating a short-maturity and a long-maturity interest rate.

[3]For example, the Diebold et al. (2005) model requires the estimation of 66 parameters (via the application of the Kalman filter and maximum likelihood) for the 20 variables it uses (i.e a price variable, a real activity variable, a monetary policy variable, and interest rates for 17 maturities). Similarly, the Ang and Piazzesi (2003) model requires the estimation of 18 parameters (via the multistep application of maximum likelihood) for the ten variables it uses (i.e three price series, four indicators of real activity, and interest rates for five maturities).

[4]Bank for International Settlements (1999) provides a survey of the use of NS model by central banks. Recent examples of the application of NS models include Fang and Muljono (2003), Diebold and Li (2005), and Diebold et al. (2005).

(1985a), and then reinterprets that model into macroeconomic quantities; i.e aggregate output growth and economy-wide inflation, and the steady-state values of those two quantities. Section 3 introduces the augmented NS (hereafter ANS) model, an intertemporally-consistent and arbitrage-free model of the yield curve from Krippner (2005),[5] and then explicitly relates the state variables from the ANS model to the macroeconomic quantities specified in section 2. This provides the platform for deriving discrete-time single-equation econometric relationships analogous to those used in the existing literature, but with a rigorous theoretical foundation. Section 4 uses United States data to estimate the econometric relationships derived in section 3. Section 5 concludes, and notes several potential applications of the ANS framework.

2 A generic model for the macroeconomy and the yield curve

This section develops a model of the macroeconomy and its associated yield curve. Section 2.1 specifies an augmented version of the Berardi and Esposito (1999) (hereafter BE) model, which is itself a generic multifactor version of the standard continuous-time general-equilibrium-economy model proposed by Cox et al. (1985a). Section 2.2 specifies macroeconomic quantities from the augmented BE (hereafter ABE) model, and in light of these interpretations then justifies the specification and assumptions from section 2.1.

2.1 A generic general-equilibrium-economy model of the yield curve

The ABE economy is based on J real factors of production (e.g capital, labour, etc., potentially by industry sector), each with its own associated deflator/inflation factor. The dynamics of the ABE economy are represented by $2J$ processes analogous to the Vasicek (1977) specification, i.e:

$$ds_j\left(t\right) = -\kappa_j\left[s_j\left(t\right) - \theta_j\left(t\right)\right]dt + \sigma_{1,j}dz_{1,j}\left(t\right) \tag{1}$$

where $s_j\left(t\right)$ for $j=1$ to J are the real state variables representing instantaneous growth on returns to the factors of production in the economy at time t; κ_j are positive constant mean-reversion parameters; $\theta_j\left(t\right)$ are the steady-state (i.e long-run) values of $s_j\left(t\right)$ which vary stochastically over time as $d\theta_j\left(t\right) = \sigma_{0,j}dz_{0,j}\left(t\right)$; $\sigma_{0,j}$ and $\sigma_{1,j}$ are positive constant standard deviation parameters with $\sigma_{0,j} \ll \sigma_{1,j}$; and $dz_{0,j}\left(t\right)$ are $dz_{1,j}\left(t\right)$ are independent Wiener variables under the physical (i.e non-risk neutral) measure. For $j = J+1$ to $2J$, $s_j\left(t\right)$ are the inflation state variables. BE shows that these have the form $s_{J+j}\left(t\right) = \pi_j\left(t\right) - \sigma_{j,p}^2$, where $\pi_j\left(t\right)$ is the instantaneous rate of inflation for the factor of production j, and $\sigma_{j,p}^2$ is a positive constant parameter representing the variance of instantaneous changes in the deflator j. Similarly, $\theta_{J+j}\left(t\right) = \theta_{j,\pi}\left(t\right) - \sigma_{j,p}^2$, where $\theta_{j,\pi}\left(t\right)$ is steady-state rate of inflation for the factor of production j. The parameters for the inflation state variables are analogous to the real state variables. Following BE, this article also assumes for mathematical convenience that all state variables $s_j\left(t\right)$ and their associated steady-state variables $\theta_j\left(t\right)$ are constructed from the original $2J$ state variables and $2J$ steady-state variables so that all innovations $dz_{0,j}\left(t\right)$ and $dz_{1,j}\left(t\right)$ are uncorrelated.

Given the stochastic processes specified in equation 1, appendix A derives the expected path of the short rate and the default-free forward rate curve in the economy (both instantaneous and continuously-compounding) using the Heath, Jarrow and Morton (1992) (hereafter HJM) framework. That is:

$$E_t\left[r\left(t+m\right)\right] = \sum_{j=1}^{2J}\theta_j\left(t\right) + \sum_{j=1}^{2J}\left[s_j\left(t\right) - \theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right) \tag{2}$$

where E_t is the expectations operator at time t; m (≥ 0) denotes a horizon from time t, so $t + m$ represents a future point in time; and $E_t\left[r(t+m)\right]$ is, as at time t, the expected path of the short rate as a function of horizon m. Similarly:

$$
\begin{aligned}
f(t,m) \;=\;& \sum_{j=1}^{2J}\theta_j(t) + m\cdot\sum_{j=1}^{2J}\sigma_{0,j}\rho_{0,j} - m^2\cdot\sum_{j=1}^{2J}\frac{1}{2}\sigma_{0,j}^2 \\
& + \sum_{j=1}^{2J}\left[s_j(t)-\theta_j(t)\right]\cdot\exp\left(-\kappa_j m\right) + \sum_{j=1}^{2J}\left[\sigma_{1,j}\rho_{1,j}-\sigma_{0,j}\rho_{0,j}\right]\cdot B_j(m) \\
& - \sum_{j=1}^{2J}\frac{1}{2}\left[\sigma_{1,j}^2-\sigma_{0,j}^2\right]\cdot\left[B_j(m)\right]^2
\end{aligned}
\tag{3}
$$

where $f(t,m)$ is the forward rate curve as at time t as a function of maturity m; $B_j(m) = \frac{1}{\kappa_j}\left[1-\exp\left(-\kappa_j m\right)\right]$ (the Vasicek (1977) functional form), and $\rho_{0,j}$ and $\rho_{1,j}$ are respectively the market prices of risk associated with the innovations $dz_{0,j}(t)$ and $dz_{1,j}(t)$. The market prices of risk arise in $f(t,m)$ because the ABE model is specified under the physical measure where, relative to the risk-free rolling investment in the short rate, investors will demand compensation for the risk associated with owning fixed interest securities (i.e unanticipated changes in market value imparted by the innovations $dz_{0,j}(t)$ and $dz_{1,j}(t)$ in $E_t\left[r(t+m)\right]$ and $f(t,m)$ as time evolves). The (zero-coupon continuously-compounding) interest rate curve, as at time t as a function of maturity m, is then defined as $R(t,m) = \frac{1}{m}\int_0^m f(t,m)dm$, and the market-quoted prices and yields-to-maturity of default-free coupon-bearing securities that compose the yield curve (hereafter simply called the yield curve) are defined by each security's cashflows discounted by $R(t,m)$.

At this stage, the ABE model is a generic and theoretically-consistent economic-finance model. That is, there is complete identification between the yield curve and an arbitrary number of real factors of production with their associated deflator/inflation factors, and the model is explicitly constructed to be dynamic, intertemporally-consistent, and arbitrage-free. However, the generic ABE model is not practically amenable. For example, a "complete" specification based on multiple factors of production by multiple industry sectors would quickly inflate to an untenable number of variables and parameters, and an approximation based on just the three real factors of production that are typically used to represent the entire economy (i.e capital, labour, and total factor productivity) would contain six state variables, six steady-state variables, and 30 parameters.[6] Even a minimal approximation based on a single factor of production would require two state variables, two steady-state variables, and ten parameters.

Alternatively, section 3 shows it is possible to use just three state variables and seven parameters to represent the generic ABE model to a precise first-order approximation. Before proceeding with that exposition, the following sub-section defines some macroeconomic quantities from the ABE model that are used subsequently in the article.

2.2 The ABE macroeconomy

Firstly, define real instantaneous output growth as $dY(t) = \sum_{j=1}^{J}s_j(t)$. That is, the sum of instantaneous growth on returns to the factors of production in the economy equals instantaneous income growth, which equals instantaneous output growth (given the economy is in continuous equilibrium). Secondly, define real instantaneous steady-state (i.e potential) output growth at time

---

[6]That is, each real state variable has associated inflation state variable. Each state variable then requires the parameters κ_j, $\sigma_{1,j}$, $\rho_{1,j}$, and the steady-state variable $\theta_j(t)$, and the latter requires the parameters $\sigma_{0,j}$, $\rho_{0,j}$. Note that this is implicitly after any orthogonalisation of the original state variables; allowing for covariances between the original state variables would require further parametrisation.

t as $dY^{*}(t) = \sum_{j=1}^{J} \theta_{j}(t)$. That is, if the returns to the factors of production are all growing at their steady-state values, then output must be growing at its steady-state value. Thirdly, define an economy-wide inflation state variable as the sum of all inflation state variables, i.e $dP(t) = \sum_{j=J+1}^{2J} s_{j}(t)$, and finally define an economy-wide steady-state inflation variable as the sum of all steady-state inflation variables, i.e $dP^{*}(t) = \sum_{j=J+1}^{2J} \theta_{j}(t)$.

These interpretations justify the ABE model specification and assumptions in section 2.1 from a macroeconomic perspective. That is, time-varying $\theta_{j}(t)$ values allow steady-state output growth and inflation to vary over time, and Gaussian innovations allow output growth and inflation to take on negative values, which are properties consistent with realised historical macroeconomic data.[7] Similarly, the orthogonalisation assumed in the construction of the ABE model implicitly (and realistically) allows for empirical covariances between inflation, output growth, and their steady-state values.

Conversely, the BE model assumes that $\theta_{j}(t)$ are constant parameters, which would result in constant steady-state output growth. The BE model can also be specified with Cox, Ingersoll and Ross (1985b) dynamics (i.e innovations of $\sqrt{s_{j}(t)} \cdot dz_{1,j}(t)$ in equation 1), but that would prohibit output growth and inflation from becoming negative. The BE model also requires additional assumptions for the single inflation state variable it uses.[8]

3 The ABE model and the ANS model of the yield curve

This section relates the ABE model of the yield curve from section 2 to the ANS model of the yield curve from Krippner (2005), which is itself a theoretically-consistent (i.e intertemporally-consistent and arbitrage-free) version of the NS model. Section 3.1 outlines the theoretical elements of the ANS model essential to this article and then illustrates the application of the ANS model in practice. Section 3.2 explicitly relates the state variables of the ANS model to the state variables and the macroeconomic quantities from the ABE model, and section 3.3 derives the discrete-time single-equation econometric relationships implied by the ANS framework.

3.1 The ANS model of the yield curve

The ANS model is based on the following specification for the expected path of the short rate:

$$E_{t}\left[r(t+m)\right] = \sum_{n=1}^{3} \lambda_{n}(t) \cdot g_{n}(\phi, m) \tag{4}$$

where $E_{t}\left[r(t+m)\right]$ is the expected path of the short rate as at time t as a function of horizon m; and $\lambda_{n}(t)$ are the three state variables that are associated with the three time-invariant functions of maturity $g_{n}(\phi, m)$ taken from the NS model. The latter are defined as $g_{1}(\phi, m) = 1$, $g_{2}(\phi, m) = -\exp(-\phi m)$, and $g_{3}(\phi, m) = -\exp(-\phi m)(-2\phi m + 1)$, where ϕ is a positive constant parameter that governs the rate of exponential decay. Figure 1 illustrates these functions, which are named the Level, Slope, and Bow modes based on their intuitive shapes.

---

[7]From a financial perspective, Gaussian innovations imply that interest rates have a non-zero probability of becoming negative. This can safely be ignored in practice (as is often done when Vasicek (1977) models are used) unless interest rates are already materially close to zero. Alternatively, a reflecting boundary at zero could be imposed, as in Goldstein and Keirstead (1997), but that is well beyond the scope of this article.

[8]Specifically, BE assumes that innovations in inflation are independent of innovations in the original real state variables, and that $\kappa_{\pi} \simeq 0$ (to allow for inflation persistence and consistency with the Fisher hypothesis). Incidentally, these assumptions result in the BE model with Gaussian dynamics being a special case of the ABE model, and therefore allows the BE model to be related directly to the ANS model in section 3. That is, setting $\theta_{j}(t) = \theta_{j}$ and $\sigma_{0,j} = 0$ in equation 3 recovers the BE expression for $f(t, m)$. Then, by setting $\kappa_{\pi} = 0$ and noting that $\lim_{\kappa_{\pi} \to 0} B_{1}(m) = m$ and $\lim_{\kappa_{\pi} \to 0} [B_{1}(m)]^{2} = m^{2}$ (by L'Hôpital's rule), the inflation component of $E_{t}\left[r(t+m)\right]$ and $f(t, m)$ in the BE model may be related precisely to the Level component of $E_{t}\left[r(t+m)\right]$ and $f(t, m)$ in the ANS model.

[Figure 1 here]

The augmentation of the ANS model relative to the NS model is to explicitly specify the stochastic dynamics for the state variables $\lambda_n(t)$ under the physical measure. This allows the derivation of an intertemporally-consistent and arbitrage-free model of the yield curve while maintaining the essence of the NS approach, i.e the yield curve at any point in time is still summarised by three estimated coefficients. Specifically, the ANS model assumes constant market prices of risk, denoted ρ_n, and independent innovations $\sigma_n \cdot g_n(\phi, m) \cdot dW_n(t)$ for each state variable $\lambda_n(t)$, where σ_n are positive constant standard deviation parameters and $dW_n(t)$ are independent Wiener variables under the physical measure. As detailed in Krippner (2005), applying the HJM framework to the components of $E_t[r(t+m)]$ under these specified dynamics results in the following forward rate curve:

$$f(t,m) = \sigma_1\rho_1 m + \sum_{n=1}^{3} \beta_n(t) \cdot g_n(\phi,m) - \sum_{n=1}^{3} \sigma_n^2 \cdot h_n(\phi,m) \tag{5}$$

where $\beta_n(t) = \gamma_n + \lambda_n(t)$ (γ_n are constant term premia parameters derived as $\gamma_1 = \frac{1}{\phi}(-\sigma_2\rho_2 + \sigma_3\rho_3)$, $\gamma_2 = \frac{1}{\phi}(-\sigma_2\rho_2 - 2\sigma_3\rho_3)$, $\gamma_3 = \frac{1}{\phi}\sigma_3\rho_3$); and $h_n(\phi,m)$ are time-invariant functions of maturity derived as $h_1(\phi,m) = \frac{1}{2}m^2$, $h_2(\phi,m) = \frac{1}{2\phi^2}[1 - \exp(-\phi m)]^2$, $h_3(\phi,m) = \frac{1}{2\phi^2}[1 - \exp(-\phi m) - 2m\phi\exp(-\phi m)]^2$.

Appendix C of Krippner (2005) details how, analogous to the estimation of NS models, the ANS coefficients $\beta_n(t)$ at any point in time may be estimated by "fitting" the yield curve data observed at that point in time, and how the parameters ϕ, ρ_1, σ_1, σ_2, and σ_3 may be estimated over an appropriate historical period. Anticipating the complete discussion of the monthly yield curve data in the empirical application of section 4.1, figure 2 illustrates an example of estimating the ANS model by "fitting" the yield curve data for June 2004. The following section explains why is valid to ignore the estimated residuals, which leaves just the estimates of the Level, Slope, and Bow coefficients as at June 2004, i.e β_1 (Jun-2004), β_2 (Jun-2004), and β_3 (Jun-2004), as the essential output. That is, when applied to the time-invariant modes $g_n(\phi,m)$, those June 2004 ANS coefficients define the expected path of the short rate as at June 2004 to within a time-invariant term premium function $\sum_{n=1}^{3} \gamma_n \cdot g_n(\phi,m)$; i.e $\sum_{n=1}^{3}[\beta_n(\text{Jun-2004}) - \gamma_n] \cdot g_n(\phi,m) = \sum_{n=1}^{3} \lambda_n(\text{Jun-2004}) \cdot g_n(\phi,m) = E_{\text{Jun-2004}}[r(\text{Jun-2004} + m)]$.

[Figure 2 here]

Each observation of yield curve data gives an associated estimate of the ANS Level, Slope, and Bow coefficients. Hence, any time series of yield curve observations may be processed into time series of Level, Slope, and Bow coefficients, i.e $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$. Figure 3 illustrates the time series of two of the seven yields used to define the yield curve at each point in time, and figure 4 plots the three time series of estimated ANS coefficients obtained using the full sample of yield curve data.

[Figure 3 here], [Figure 4 here]

3.2 Relating the ANS model to the ABE model

Comparing equations 2 and 4, the first apparent correspondence is between $\lambda_1(t)$ and $\sum_{j=1}^{2J} \theta_j(t)$, or $dP^*(t) + dY^*(t)$ using the macroeconomic quantities from section 2.1. That is, $dP^*(t) + dY^*(t) = \lambda_1(t) \cdot g_1(\phi,m) = \lambda_1(t)$, and so substituting $\lambda_1(t) = \beta_1(t) - \gamma_1$ gives:

$$\beta_1(t) = \gamma_1 + dP^*(t) + dY^*(t) \tag{6}$$

Hence, the Level coefficient from the ANS model at time t is composed of a constant term premium component γ_1, and the economy-wide steady-state inflation variable plus steady-state output growth at time t. Because both the ANS and ABE models are specified with Gaussian dynamics,

the innvotions in the ANS Level coefficient correspond precisely to the ABE steady-state inno-vations, i.e $\sigma_1 dW_n(t) = \sum_{j=1}^{2J} \sigma_{0,j} dz_{0,j}(t)$. In addition, the component of the ANS forward rate curve associated with the Level coefficient and its dynamics corresponds precisely to the component of the ABE forward rate curve associated with the steady-state components and their dynamics. That is, the Level component of the ANS forward rate equation 5 is $\sigma_1\rho_1 m + \beta_n(t) - \sigma_1^2 \cdot \frac{1}{2}m^2$ which corresponds to steady-state component of the ABE forward rate equation (the first line of equation 3) with $\sigma_1\rho_1 = \sum_{j=1}^{2J} \sigma_{0,j}\rho_{0,j}$ and $\sigma_1^2 = \sum_{j=1}^{2J} \sigma_{0,j}^2$. This correspondence between forward curves ensures that when yield curve data observed at time t is "fitted" using the ANS model, the Level coefficient $\beta_1(t)$ will be a consistent estimate, to within a constant γ_1, of the sum of the steady-state components of the ABE model as at time t.

The "remainder" of the yield curve as estimated by the ANS model (i.e the Slope and Bow components, and the residuals from the yield curve estimation) will therefore relate to the non-steady-state components of the ABE model. Indeed, appendix B.1 proves that the ANS Slope and Bow components are a precise first-order approximation to the non-steady-state components of the ABE model with an arbitrary number of state variables, and the latter may also be expressed as the expected values of the ABE macroeconomic quantities from section 2.1, i.e:

$$-\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m) = -E_t[dP(t+m) + dY(t+m)$$

$$-dP^*(t+m) - dY^*(t+m)] \tag{7a}$$

$$\simeq \sum_{n=2}^{3}[\beta_n(t) - \gamma_n] \cdot g_n(\phi, m) \tag{7b}$$

where E_t is the expectations operator at time t, and $dP(t+m)$, $dY(t+m)$, $dP^*(t+m)$, and $dY^*(t+m)$ are respectively the economy-wide inflation state variable, real output growth, the economy-wide steady-state inflation variable, and steady-state output growth, all at the future time $t+m$. The approximation in equation 7 can also been seen as a reduction in dimensionality that is commonly undertaken using latent factor models for the yield curve (e.g see Ang and Piazzesi (2003) and Diebold et al. (2005)), but with an underlying theoretical structure; i.e the two time-varying coefficients $\beta_2(t)$ and $\beta_3(t)$ applied to the factors $g_2(\phi, m)$ and $g_3(\phi, m)$ are being used to represent the expected evolution of $2J$ state variables relative to their steady-state values, i.e $\sum_{j=1}^{J}[s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m)$, each with a different mean-reversion parameter κ_j. Appendix B.1 also shows that this precise first-order approximation also carries through to the innovations and forward rate curves.

3.3 Econometric relationships for the ANS model coefficients, and inflation and output growth

For econometric estimation, the continuous-time relationships noted in the previous section need to be expressed as annualised discrete-time relationships. The elements of equation 6 are all con-temporaneous, and so its discrete-time version is simply:

$$\beta_{1,t} - \Delta P_t^* - \Delta Y_t^* = \alpha^* + \varepsilon_t^* \tag{8}$$

where $\beta_{1,t}$ is the estimated Level coefficient, ΔP_t^* is steady-state inflation, and ΔY_t^* is steady-state output growth, all at time t. The constant parameter α^* captures the term premium component γ_1 and the parameters $\sigma_{j,p}^2$ noted in section 2.1. Equation 8 is therefore a (1,-1) cointegrating relationship between $\beta_{1,t}$ and $\Delta P_t^* + \Delta Y_t^*$, and ε_t^* represents a time series of estimated residuals that should be stationary. Note that appendix B.2 proves that all of the data in equation 8 should be Gaussian processes, meaning OLS estimation and standard unit root tests are applicable.

Equation 7 is an intertemporal relationship, so its discrete-time version requires the appropriate notation for both time and horizon. Hence, denote a forward interval from time t by $t + T_1, t + T_2$ where T_1 (≥ 0) and T_2 $(> T_1)$ are both constants. For notational convenience, then define the quantity "expected relative nominal output growth" (i.e the expected change in instantaneous nominal output growth relative to steady-state nominal output growth) as $E_t\left[dX\left(t+m\right)\right] = E_t\left[dP\left(t+m\right) + dY\left(t+m\right) - dP^*\left(t+m\right) - dY^*\left(t+m\right)\right]$. The discrete-time measure of expected relative nominal output growth over the forward horizon $t + T_1, t + T_2$, denoted $E_t\left[\Delta X_{t+T_1,t+T_2}\right]$, is then the average of $E_t\left[dX\left(t+m\right)\right]$ over the forward interval. For example, $E_t\left[\Delta X_{t,t+1}\right]$ is expected relative nominal output growth over the following year (i.e now to one year from now), and $E_t\left[\Delta X_{t+0.5,t+0.75}\right]$ is expected relative nominal output growth over the period two quarters from now to three quarters from now.

The corresponding averages of the ANS model terms from equation 7, i.e $-\sum_{n=1}^{3}\left[\beta_n\left(t\right) - \gamma_n\right] \cdot g_n(\phi, m)$, may be calculated by via integration by maturity over the forward interval, i.e:

$$\frac{-1}{T_2 - T_1}\int_{t+T_1}^{t+T_2}\left[\sum_{n=2}^{3}\left[\beta_n\left(t\right) - \gamma_n\right] \cdot g_n(\phi, m)\right]dm = -\sum_{n=2}^{3}\gamma_n \cdot q_n\left(T_1, T_2\right) + \sum_{n=2}^{3}\beta_n\left(t\right) \cdot q_n\left(T_1, T_2\right) \quad (9)$$

where $q_n\left(T_1, T_2\right) = \frac{-1}{T_2 - T_1}\int_{T_1}^{T_2}g_n(\phi, m)dm$. The two calculations required are:

$$q_2\left(T_1, T_2\right) = \frac{-1}{\phi\left(T_2 - T_1\right)}\left[\exp\left(-\phi T_2\right) - \exp\left(-\phi T_1\right)\right] \quad (10a)$$

$$q_3\left(T_1, T_2\right) = \frac{-1}{\phi\left(T_2 - T_1\right)}\left[\exp\left(-\phi T_2\right)\left(-2\phi T_2 - 1\right) - \exp\left(-\phi T_1\right)\left(-2\phi T_1 - 1\right)\right] \quad (10b)$$

and table 3 contains the values of $q_2\left(T_1, T_2\right)$ and $q_3\left(T_1, T_2\right)$ that correspond to the forward horizons tested in the empirical work. For example, using the ANS Slope and Bow coefficients estimated for June 2004 (from figure 2), expected relative nominal output growth from June 2004 to June 2005 excluding any term premium estimate is $6.80\% \times 0.61 - 2.09\% \times 0.07 = 4.03\%$. Similarly, expected annualised relative nominal output growth between December 2004 to March 2005 is $6.80\% \times 0.51 - 2.09\% \times -0.17 = 3.84\%$. Because the values of γ_2 and γ_3 are constant in the ANS model, for each forward horizon the quantity $-\sum_{n=2}^{N}\gamma_n \cdot q_n\left(T_1, T_2\right)$ will be a constant, which may be denoted as α_{0,T_1,T_2}. Hence, the discrete-time single-equation relationship for each forward horizon is:

$$E_t\left[\Delta X_{t+T_1,t+T_2}\right] = \alpha_{0,T_1,T_2} + \alpha_{1,T_1,T_2} \cdot \sum_{n=2}^{3}\beta_n\left(t\right) \cdot q_n\left(T_1, T_2\right) + \varepsilon_{t,T_1,T_2} \quad (11)$$

Note that the coefficient α_{1,T_1,T_2} should equal 1 for each forward horizon, because the intertemporal profile of $E_t\left[\Delta X_{t+T_1,t+T_2}\right]$ is already embedded in $q_n\left(T_1, T_2\right)$. As standard in the existing literature (e.g see Estrella et al. (2003)), the estimation of equation 11 proceeds by substituting realised $\Delta X_{t+T_1,t+T_2}$ data for $E_t\left[\Delta X_{t+T_1,t+T_2}\right]$. Appendix B.2 proves that all of the data in equation 11 should be Gaussian processes, meaning OLS estimation is applicable. The Newey-West technique may also be used to correct the estimated standard errors for the effect of moving-average serial correlation induced in ε_{t,T_1,T_2}, which occurs whenever the forward horizon exceeds the frequency of the data (this technique is standard in the existing literature; e.g see Estrella et al. (2003)).

Note that the ANS framework above has been developed for the nominal yield curve, which relates directly to the empirical application in the following section. Of course, the ANS model linked explicitly to the real ABE model (obtained by simply omitting the deflator/inflation state variables, the associated steady-state variables, and the deflator/inflation parameters) would produce an analogous real ANS framework applicable to inflation-indexed yield curve data.

4 An empirical application to US data

This section tests the predictions of the ANS model framework empirically using US data. To make the results directly comparable to the existing literature and to allow for the longest period of estimation, the empirical analysis is undertaken in-sample only using standard published data. Section 4.1 outlines that data, section 4.2 discusses the results from estimating the predicted yield curve/inflation relationships, and 4.3 discuss the results from estimating the predicted yield curve/output growth relationships.

4.1 Description of the yield curve, inflation, and output data

The interest rate data used in the empirical application are obtained from the online Federal Reserve Economic Database (hereafter FRED) on the Federal Reserve Bank of St Louis website. The specific series are the monthly averages of the federal funds rate, the 3-month Treasury bill rate, and the 1-year, 3-year, 5-year, 10-year, and 20-year or 30-year constant-maturity bond rates.[9] The sample period is July 1954 (the first month federal funds rate data is available) to June 2005 (the last month available at the time of the analysis), giving 612 monthly observations of the yield curve. Figure 4 has already illustrated the monthly time series of ANS Level, Slope, and Bow coefficients derived from the yield curve data, and taking the last month of each quarter provides the relevant quarterly data for this article. Note that the ANS coefficients are already on an annualised basis, given that they are estimated from yield curve data expressed on an annualised basis.

The analysis also requires data for steady-state inflation, steady-state output growth, and nominal output growth relative to its steady-state value. These data are not measured directly, and so proxies are necessarily required. The primary proxy for economy-wide steady-state inflation is chosen as inflation in the GDP deflator (hereafter IGD). This choice means that nominal output growth relative to its steady-state value equals real output growth relative to its steady-state value, i.e $\Delta X_{t+T_1,t+T_2} = \Delta Y_{t+T_1,t+T_2} - \Delta Y^*_{t+T_1,t+T_2}$, for which there is standard data available. Specifically, the proxy for real output growth is real GDP growth, and the proxy for steady-state output growth is Congressional Budget Office potential GDP growth (hereafter CBO ΔY^*_t; see Congressional Budget Office (2001) for calculation details). Inflation in the personal consumption expenditure deflator, including and excluding food and energy (hereafter PCE and PCEX respectively), are also tested as alternative proxies for steady-state inflation when testing for the cointegration implied by equation 8.[10] All index levels for the series mentioned are available from the FRED on a quarterly basis, and the inflation and growth data are calculated as changes in the logarithm of those levels.

Equations 8 and 11 are estimated using both annualised quarterly data, and quarterly annual data; the latter for comparability to the existing literature. The existing literature also forecasts GDP growth, rather than GDP growth relative to potential GDP growth. Using a constant estimate of potential output growth (i.e $\Delta Y^*_t = 3.31\%$, which is the average of annualised quarterly GDP growth over the entire sample), instead of the time-varying CBO ΔY^*_t obtains $\Delta X_{t+T_1,t+T_2}$ data equivalent to GDP growth to within a constant.

To allow a visual inspection of some of the relationships to be estimated, figure 5 plots the time series of the ANS Level coefficient and the annualised quarterly IGD plus annualised quarterly CBO ΔY^*_t data, and figure 6 plots the difference between the Level coefficient and IGD plus CBO ΔY^*_t. Figure 7 illustrates the annual GDP growth, the annual CBO ΔY^*_t, and the constant $\Delta Y^*_t = 3.31\%$ data that are used to calculate the annual $\Delta X_{t+T_1,t+T_2}$ data subsequently used in the estimation of

---

[9]20-year data is unavailable from January 1987 to September 1993, and so 30-year data (with a 30-year maturity) is used during this period for the estimation of the ANS model.

[10]Surveyed long-term, or even short-term, CPI inflation expectations would arguably make superior proxies for steady-state inflation. However, the availability of that data is limited; e.g 10-year inflation expectations from the Philidelphia Federal Reserve website are only available from 1991, and Michigan year-ahead inflation expectations from the FRED are only available from 1978. An alternative proxy for steady-state inflation might be a suitable trend extracted from the historical data.

equation 11. Figure 8 plots $\Delta X_{t,t+1}$ based on CBO ΔY_t^*, and the corresponding predicted values of $\Delta X_{t,t+1}$ using the ANS Slope and Bow coefficients that prevailed at time t.

[Figure 5 here] [Figure 6 here], [Figure 7 here], [Figure 8 here]

Figures 5, 6, and 8 all show prima facie evidence of structural breaks in the yield curve/inflation and yield curve/output growth relationship from around the late 1970s/early 1980s. This is not surprising given the context of substantial change in the US economic and financial environment during the late-1970s to the mid-1980s. For example, a significant economic change was the Federal Reserve's Volcker-led disinflation from October 1979, and the subsequent maintenance of low inflation. A significant financial change was progressive market deregulation, including eliminating interest rate restrictions, rationalising reserve requirements, and allowing an increasing role for securitisation. Prior empirical work also mentions these reasons when documenting structural breaks between 1979:Q4 to 1984:Q1.[11]

The analysis for the full sample therefore proceeds with the inclusion of a step dummy variable D_t in equations 8 and 11, where $D_t = 0$ up to the period immediately before the breakpoint, and $D_t = 1$ from the given breakpoint. The breakpoints for the analysis in this article are 1982:Q1 for inflation and 1984:Q1 for output, which were selected to be consistent with the prior empirical work noted above. The analysis is also undertaken for the two sub-samples pre-1979:Q4 and post-1984:Q1, which excludes all of the breakpoints referenced in prior empirical work.

4.2 The ANS Level coefficient and inflation

Tables 1 and 2 contain the test results for cointegration, as as implied by equation 8, between the ANS Level coefficient and the annualised quarterly and annual measures of inflation plus CBO potential GDP growth.[12] Both sets of results are moderately supportive of the hypothesis of cointegration over the whole sample and the two sub-samples. Specifically, the test statistics in the top half of the table typically do not reject the unit root hypothesis,[13] but the Level coefficient less the measures of inflation with or without potential growth added typically do reject the unit root hypothesis. Consistent with the prior discussion on structural change, the cointegration results over the whole sample are stronger when the structural change dummy variable is included. Interestingly, the results are also better when CBO ΔY_t^* is ignored (which is equivalent to replacing CBO ΔY_t^* with the constant $\Delta Y_t^* = 3.31\%$), and/or measures of consumption inflation are used as proxies for steady-state inflation. These observations suggest that long-maturity yields may be more responsive to movements in consumption inflation measures, rather than economy-wide inflation and/or variations in steady-state output growth.

[Table 1 here], [Table 2 here]

That said, any conclusions must remain tentative given what is essentially modest and variable evidence for cointegration. This might be because current inflation and potential output growth

---

[11] For example, based on statistical tests for unknown breakpoints, Estrella et al. (2003) identifies structural breaks in October 1979 and October 1982 when the yield curve is used as an indicator of future inflation, and in September 1983 when the yield curve is used as an indicator of future output. Using a similar technique, Aïssa and Jouini (2003) documents a break in the inflation process in June 1982, and Lai (2004) suggests a break between mid-1980 and early-1981. Jardet (2004) documents a break in the yield curve/output relationship in March 1984, and the yield curve forecasting application of Krippner (2005) implies a structural break in yield curve term premia between September 1979 and October 1982.

[12] For consistency, all quarterly results use one lag for the augmented Dickey-Fuller tests and a window of one for the Phillips-Perron tests, and all annual results use four lags and a window of four (to allow for the expected MA(3) serial correlation plus one). The results using optimal lag and window selection were similar, but implausibly long lag lengths were occasionally selected.

[13] The quarterly measures of inflation often produce materially negative test statistics, but MacKinnon (1996) p. 615 notes that test statistics on the annual measures of inflation are more reliable (essentially because unobservable measurement errors in inflation data over short intervals tend to bias the unit root test statistics downward, but that bias fades over longer intervals). Of course, any downward bias in the unit root tests for inflation will also translate into the cointegration tests, but the latter test statistics are typically of a larger magnitude than the unit root test statistics on inflation itself.

are not good proxies for their steady-state counterparts, but an alternative explanation is that the variance of relative changes in the deflators (i.e the parameters $\sigma_{j,p}^2$ noted in section 2.1) might vary over time, as might the risk premia related to steady-state inflation and/or steady-state growth. Indeed, if the combination of those quantities over time are represented by the time series $\beta_{1,t} - \Delta P_t^* - \Delta Y_t^*$, then figure 6 shows four distinct levels: i.e a low (but variable) level up to the late-1970s/early-1980s; a peak level from the early-1980s to 1986; a moderate level from 1986 to 1998; and a return to a relatively low level (i.e consistent with pre-1979 levels) from 1998. It would be intriguing to formally test for structural breaks in the time series $\beta_{1,t} - \Delta P_t^* - \Delta Y_t^*$, and to see how those breaks correspond to changes in the economic and financial environments that prevailed at the time. However, that investigation is beyond the scope of this article, and so will be explored in future work.[14]

4.3   The ANS Slope and Bow coefficients and output growth

Table 3 contains the results from estimating equation 11 over the full sample, using the dummy variable with the 1984:Q1 breakpoint and $\Delta X_{t+T_1,t+T_2}$ based on CBO ΔY_t^*.[15]   The first point of note is that the yield curve has explanatory power for $\Delta X_{t+T_1,t+T_2}$ over short and medium horizons. That is, the coefficients α_{1,T_1,T_2} are highly significant and positive for forward horizons up to one year, become insignificant while remaining positive through the second year (although the coefficient is significant on an annual basis), but become insignificant and negative for most forward horizons over two years. In addition, the coefficients α_{1,T_1,T_2} are insignificantly different from the theoretical value of 1, except for the marginal rejection of that hypothesis for the 2.25 to 2.5 year, and the 2.5 to 2.75 year horizons. This indicates that the ANS framework provides a gauge of the profile (i.e the timing and magnitude) of the future changes in output growth relative to potential output growth. Both the constant and the dummy variable are highly significant for short horizons (and remain consistently signed but insignificant after that), suggesting that a term premium existed before the structural break, and became larger in magnitude after the structural break. The negative value of both coefficients is consistent with positive term premia; i.e the yield curve would persistently over-forecast $\Delta X_{t+T_1,t+T_2}$, so a negative adjustment is required to remove that persistent bias.

[Table 3 here]

Table 4 contains the results for equation 11 estimated over each sub-sample, and this provides an interesting insight into the results for the entire sample. That is, up to 1979:Q3 the shape of the yield curve was best at predicting $\Delta X_{t+T_1,t+T_2}$ over short forward horizons, although it tended to under-predict those changes. Conversely, beyond 1984:Q1 the shape of the yield curve was best at predicting $\Delta X_{t+T_1,t+T_2}$ over medium forward horizons; while remaining useful for short horizons, it tended to over-predict $\Delta X_{t+T_1,t+T_2}$. The combination of these sub-sample results evidently offset to give estimates of the coefficients α_{1,T_1,T_2} that are close to unity over the full sample.

[Table 4 here]

Tables 5 and 6 contain the results for estimating equation 11 using $\Delta X_{t+T_1,t+T_2}$ based on $\Delta Y_t^* = 3.31\%$. These estimations are now directly analogous to the regressions of GDP growth on lagged yield curve spreads from the existing literature, and therefore provide some insights into those prior results. Firstly, the existing literature finds the explanatory power of the regressions are highest for short forward horizons and fade quickly past forward horizons of one year (e.g see Hamilton and Kim (2002) table 2). Equation 11 shows this is to be expected, given that the $\sum_{n=2}^{3} \beta_n(t) \cdot q_n(T_1, T_2)$ "signal" decreases (due to the falling magnitudes of $q_n(T_1, T_2)$) while the

---

[14]Buraschi and Jiltsov (2005) provide evidence for a time-varying risk premium on inflation using an arbitrage-free structural model of the macroeconomy and yield curve. However, the more parimonious ANS framework should prove more amenable to investigating such phenomena.

[15]The Newey-West window used in each estimation is the number of quarters to T_2 less 1. This allows for the induced serial correlation expected in theory for both the annualised quarterly and the annual data. For example, both $\Delta X_{t,t+1}$ and $\Delta X_{t+0.75,t+1}$ data will induce MA(3) serial correlation into ε_{t,T_1,T_2}.

ε_{t,T_1,T_2} "noise" increases (due to the aggregation of more expectational surprises) as the forward horizon lengthens. Secondly, the ANS framework results show better explanatory power using the CBO ΔY_t^* compared to $\Delta Y_t^* = 3.31\%$, suggesting that the results in the existing literature might be improved by allowing for time-varying potential output growth.

[Table 5 here], [Table 6 here]

5 Conclusions and potential applications

This article develops the ANS framework; a theoretically-consistent and easy–to-apply foundation for interpreting, investigating, and monitoring the relationships between the yield curve, output, and inflation. The empirical results based on US data are consistent with the framework's predictions; i.e the estimated long-maturity level of the yield curve given by the Level coefficient in the ANS model is cointegrated with steady-state inflation plus steady-state output growth, and the shape of the yield curve given by the Slope and Bow coefficients in the ANS model corresponds to the profile (i.e the timing and magnitude) of future output growth relative to its steady-state value. The estimation techniques used within the ANS framework are routine, so its practical application should be well-suited to researchers and market practitioners.[16]

An obvious practical use for the ANS framework is to extract implied market expectations of inflation and output growth directly from the yield curve, and to track changes in those expectations over time (particularly to gauge the response to economic and financial events such as data releases or monetary policy decisions). Specifically, the ANS model applied to the nominal yield curve implies the market's determination of steady-state nominal output growth and the profile of future nominal output growth. The real components of those latter quantities are implied by applying the ANS framework to the inflation-indexed (i.e real) yield curve, thereby determining the implied inflation components. However, time-varying term premia (for which this article provides some preliminary evidence) also need to be considered when extracting market expectations implied by the yield curve.

The central bank can use the information from the ANS framework as inputs into its own economic assessments, and its formulation, implementation, and communication of monetary policy with respect to its policy targets. For example, a rise in implied steady-state inflation to above the stated inflation target might add to a case for tightening monetary policy. Market practitioners should also find the information useful; indeed, the ANS framework applied in reverse should provide a rigorous foundation for converting non-consensus macroeconomic views into optimal trading positions on the level and shape of the yield curve. Similarly, another potential application of the ANS framework is to use the yield curve to value and hedge some of the macroeconomic derivatives that have been suggested by Shiller (1993 and 2003), and that have been provided to the market over recent years (e.g see Frankel and O'Neill (2002), Chicago Mercentile Exchange (2005), and Goldman Sachs (2005)).

A Derivation of the ABE forward rate curve

This appendix derives the ABE forward rate curve via the HJM framework. It proceeds in four sections: (1) outlining the relevant notation and results from the HJM framework; (2) calculating the expected path of the short rate for the ABE model; and (3) calculating the effect that the market prices of risk and volatility in the ABE model coefficients have on the shape of the forward rate curve, thereby obtaining the ABE model of the forward rate curve.

---

[16]Indeed, as noted in Bank for International Settlements (1999), central banks are already frequent users of Nelson and Siegel (1987) models, which can easily be modified into the theoretically-consistent ANS model.

A.1 The HJM framework

HJM specifies the relationship between the instantaneous forward rate curve and the instantaneous short rate under the physical measure as:[17]

$$
\begin{aligned}
r\left(t+m\right) \;=\;\; & f\left(t,m\right)+\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(v,m\right)\left\{\int_{s}^{m}\sigma_{n}\left(v,u\right)du\right\}dv \\
& -\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(v,m\right)\rho_{n}dv+\sum_{n=1}^{N}\int_{t}^{t+m}\sigma_{n}\left(v,m\right)dW_{n}\left(v\right)
\end{aligned}
\tag{12}
$$

where $r\left(t+m\right)$ is the short rate at time $t+m$; $f\left(t,m\right)$ is the forward rate curve at time t as a function of maturity m $\left(m\geq0\right)$; N is the number of independent stochastic processes that impart instantaneous random changes to the forward rate curve; $\sigma_{n}\left(v,m\right)$ is the volatility function for the process n; θ_{n} is the market price of risk for the process n; $dW_{n}\left(v\right)$ are independent Wiener variables under the physical measure; and u and v are dummy integration variables. The first two integrals in equation 12 have been written with limits 0 and m (i.e independent of t) because the market prices of risk from the ABE model are constant, and the volatility functions are time-variant functions of maturity. The third integral retains time dependence via the paths of the Wiener processes.

Applying the expectations operator as at time t to equation 12 and rearranging provides a relationship that will hold at any point in time, i.e:

$$
f\left(t,m\right)=E_{t}\left[r\left(t+m\right)\right]+\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(v,m\right)\rho_{n}dv-\sum_{n=1}^{N}\int_{0}^{m}\sigma_{n}\left(v,m\right)\left\{\int_{s}^{m}\sigma_{n}\left(v,u\right)du\right\}dv
\tag{13}
$$

where $E_{t}\left[r\left(t+m\right)\right]$ is the expected path of the short rate at time t as a function of horizon m, and the expectation of the stochastic term in equation 12 is zero (see Ross (1997) pp. 541-542).

A.2 The ABE expected path of the short rate

Following BE and Cox et al. (1985b), the nominal short rate at any given time is the summation of state variables $s_{j}\left(t\right)$, i.e $r\left(t\right)=\sum_{j=1}^{2J}s_{j}\left(t\right)$. This equality holds at all points in time, and so $E_{t}\left[r(t+m)\right]=\sum_{j=1}^{2J}E_{t}\left[s_{j}\left(t+m\right)\right]$, where $E_{t}\left[s_{j}\left(t+m\right)\right]$ are the expected values of the state variables j, all as at time t as a function of horizon m.

$E_{t}\left[s_{j}\left(t+m\right)\right]$ may be calculated by applying the expectations operation E_{t} to equation 1 and noting that $E_{t}\left[\theta_{j}\left(t+m\right)\right]=\theta_{j}\left(t\right)$; hence $E_{t}\left[ds_{j}\left(t+m\right)\right]=-\kappa_{j}\left\{E_{t}\left[ds_{j}\left(t+m\right)\right]-\theta_{j}\left(t\right)\right\}dm$. This ordinary differential equation in m has the solution $E_{t}\left[s_{j}\left(t+m\right)\right]=\theta_{j}\left(t\right)+A_{j}\cdot\exp\left(-\kappa_{j}m\right)$. The boundary condition at $m=0$ is $s_{j}\left(t\right)=\theta_{j}\left(t\right)+A_{j}$, so $A_{j}=s_{j}\left(t\right)-\theta_{j}\left(t\right)$, and therefore $E_{t}\left[s_{j}\left(t+m\right)\right]=\theta_{j}\left(t\right)+\left[s_{j}\left(t\right)-\theta_{j}\left(t\right)\right]\cdot\exp\left(-\kappa_{j}m\right)$. Summing this result across all $2J$ state variables gives $E_{t}\left[r\left(t+m\right)\right]$ as specified in equation 2.

A.3 The ABE forward rate curve

The HJM volatility functions for each component of the ABE model are determined by the stochastic innovations for each factor j applied to the components of $E_{t}\left[r(t+m)\right]$ associated with that factor. Firstly, an innovation $dz_{0,j}\left(t\right)$ will result in a parallel shift of $\sigma_{0,j}\cdot dz_{0,j}\left(t\right)$ to $E_{t}\left[r(t+m)\right]$ and $f\left(t,m\right)$ simultaneously. Therefore, the volatility function is $\sigma_{n}\left(v,m\right)=\sigma_{0,j}$ for any j, making the first equation 13 integral $\int_{0}^{m}\sigma_{0,j}\rho_{0,j}dv=\sigma_{0,j}\rho_{0,j}\cdot\left[v\right]_{0}^{m}=\sigma_{0,j}\rho_{0,j}\cdot m$, and the second equation 13 integral $\int_{0}^{m}\sigma_{0,j}\cdot\left\{\int_{v}^{m}\sigma_{0,j}du\right\}dv=\int_{0}^{m}\sigma_{0,j}\cdot\left\{\sigma_{0,j}\cdot\left[u\right]_{v}^{m}\right\}dv=\int_{0}^{m}\sigma_{0,j}^{2}\cdot\left[m-v\right]dv=\sigma_{0,j}^{2}\cdot\left[mv-\frac{v^{2}}{2}\right]_{0}^{m}=\frac{1}{2}\sigma_{0,j}^{2}\cdot m^{2}$.

---

[17]From HJM eq. 5 with the substitution of HJM eq. 25.

Secondly, an innovation $dz_{1,j}(t)$ will result in a non-parallel shift of $\sigma_{1,j} \cdot \exp(-\kappa_j m) \cdot dz_{1,j}(t)$ (i.e an exponential decay function by horizon/maturity) to $E_t[r(t+m)]$ and $f(t,m)$ simultaneously. Therefore, the volatility function is $\sigma_n(v,m) = \sigma_{1,j} \cdot \exp(-\kappa_j m)$ for any j, making the first equation 13 integral $\int_0^m \sigma_{1,j} \cdot \exp(-\kappa_j v) \cdot \rho_{1,j} dv = \sigma_{1,j} \rho_{1,j} \cdot \left[-\frac{1}{\kappa_j} \exp(-\kappa_j v) \right]_0^m = \sigma_{1,j} \rho_{1,j} \cdot B_j(m)$ where $\frac{1}{\kappa_j}[1 - \exp(-\kappa_j m)]$. The second equation 13 integral is calculated in two steps, i.e: $\int_v^m \sigma_n(v,u) du$ $= \int_s^m \sigma_{1,j} \cdot \exp(-\kappa_j[u-v]) du = \sigma_{1,j} \cdot \left[-\frac{1}{\kappa_j} \exp(-\kappa_j[u-v]) \right]_v^m = \frac{\sigma_{1,j}}{\kappa_j} \cdot [1 - \exp(-\kappa_j[m-v])]$. Then $\int_0^m \sigma_n(v,m) \left\{ \int_v^m \sigma_n(v,u) du \right\} dv$ is calculated as: $\int_0^m \sigma_{1,j} \cdot \exp(-\kappa_j[m-v]) \cdot \frac{\sigma_{1,j}}{\kappa_j} \cdot [1 - \exp(-\kappa_j[m-v])] dv = \frac{\sigma_{1,j}^2}{\kappa_j} \cdot \int_0^m [\exp(-\kappa_j[m-v]) - \exp(-2\kappa_j[m-v])] dv = \frac{\sigma_{1,j}^2}{\kappa_j^2} \cdot [\exp(-\kappa_j[m-v]) - \frac{1}{2}\exp(-2\kappa_j[m-v])]_0^m = \frac{\sigma_{1,j}^2}{\kappa_j^2} \cdot [1 - \exp(-\kappa_j m) - \frac{1}{2} + \frac{1}{2}\exp(-2\kappa_j m)] = \frac{\sigma_{1,j}^2}{2\kappa_j^2} \cdot [1 - \exp(-\kappa_j m) + \exp(-2\kappa_j m)] = \frac{\sigma_{1,j}^2}{2\kappa_j^2} \cdot [1 - \exp(-\kappa_j m)]^2 = -\frac{1}{2}\sigma_{1,j}^2 \cdot [B_j(m)]^2$

Thirdly, innovations in $dz_{0,j}(t)$ will also result in a non-parallel shift of $-\sigma_{0,j} \cdot \exp(-\kappa_j m) \cdot dz_{0,j}(t)$ to $E_t[r(t+m)]$ and $f(t,m)$ simultaneously, in addition to the parallel shift already noted earlier. The integrals for these non-parallel components follow those for $dz_{1,j}(t)$ above, giving the results $-\sigma_{0,j}\rho_{0,j} \cdot B_j(m)$ and $\frac{1}{2}\sigma_{1,j}^2 \cdot [B_j(m)]^2$.

Substituting $E_t[r(t+m)]$ from section A.2 and the calculations from this section into equation 13 gives equation 3.

B The correspondence between the ABE model and the ANS model

This appendix further establishes the correspondence between the ABE model and the ANS model. The first section shows that the non-Level components of the ANS model are a precise first-order approximation to the non-steady-state components of the generic ABE model. The second section proves that the data in equations 8 and 11 are all Gaussian under the ABE and ANS model assumptions.

B.1 Relating the non-steady-state components of the ABE model to the non-Level components of the ANS model

Using the macroeconomic quantities from section 2.1 and noting that $E_t[\theta_j(t+m)] = \theta_j(t)$ leads directly to $E_t[dP(t+m) + dY(t+m) - dP^*(t+m) - dY^*(t+m)] = E_t[dP(t+m) + dY(t+m) - dP^*(t) - dY^*(t)] = \sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m)$. Negating these terms therefore establishes the equality in equation 7a.

Regarding the approximation in equation 7b, define ϕ as a central measure of the values of κ_j for $j = 1$ to $2J$, i.e $\phi = \text{central}(\kappa_j)$ (ϕ is a positive constant, because all κ_j are positive constants). Therefore $\kappa_j = \phi(1 + \Delta_j)$, and the non-steady-state component of equation 2 may be written as $\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m) = \exp(-\phi m) \cdot \sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot \exp(-\Delta_j \phi m)$. Now write each $\exp(-\Delta_j \phi m)$ as a first-order Taylor expansion around $\Delta_j = 0$; i.e substituting $\exp(-\Delta_j \phi m) \simeq 1 - \Delta_j \phi m$ and expanding gives: $\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m) \simeq \exp(-\phi m) \cdot \sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] - \exp(-\phi m) \cdot \sum_{j=1}^{2J}[s_j(t) - \theta_j(t)]\Delta_j \cdot \phi m$. The right-hand side may be rearranged as $-\exp(-\phi m) \cdot -\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot (1 - \frac{1}{2}\Delta_j) - \exp(-\phi m) \cdot -\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)]\Delta_j \cdot \frac{1}{2}(-2\phi m + 1) = \lambda_2(t) \cdot g_2(\phi, m) + \lambda_3(t) \cdot g_3(\phi, m)$, where $\lambda_2(t) = -\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot (1 + \frac{1}{2}\Delta_j)$ and $\lambda_3(t) = -\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)]\frac{1}{2}\Delta_j$.

Hence, $\sum_{n=2}^{3}\lambda_n(t) \cdot g_n(\phi, m) \simeq -\sum_{j=1}^{2J}[s_j(t) - \theta_j(t)] \cdot \exp(-\kappa_j m)$, which upon the substitution of $\beta_n(t) - \gamma_n = \lambda_n(t)$ gives equation 7b. The non-Level component of the ANS expected path of the short rate is therefore a precise first-order approximation to the ABE expected path

of the short rate based on an arbitrary number of factors of production. Hence, ignoring the yield residuals after estimating the ANS model is equivalent to ignoring the second-order and higher terms from the Taylor expansion of the ABE model.[18]

By reference to appendix A.3, the stochastic components of $-\sum_{j=1}^{2J}\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$ are $-\sum_{j=1}^{2J}\left[\sigma_{1,j}dz_{1,j}\left(t\right)-\sigma_{0,j}dz_{0,j}\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$. Following the first-order Taylor expansion approach outlined above, the latter expression may be expressed as $\sum_{n=2}^{3}\sigma_n\cdot g_n(\phi,m)\cdot dW_n\left(t\right)$. This shows that innovations in the non-Level components of the ANS model are a first-order approximation to innovations in the the non-steady-state component of the ABE model. Finally, using these two first-order approximations, i.e $\sum_{n=2}^{3}\lambda_n\left(t\right)\cdot g_n\left(\phi,m\right)$ and $\sum_{n=2}^{3}\sigma_n\cdot g_n(\phi,m)\cdot dW_n\left(t\right)$, within the HJM framework provides the non-Level component of the ANS forward rate curve that will approximate the non-steady-state components of the ABE forward rate curve. Hence, $\sum_{n=2}^{3}\gamma_n\cdot g_n(\phi,m)\simeq\sum_{j=1}^{2J}\left[\sigma_{1,j}\rho_{1,j}-\sigma_{0,j}\rho_{0,j}\right]\cdot B_j\left(m\right)$, and $\sum_{n=2}^{3}\sigma_n^2\cdot h_2(\phi,m)\simeq-\sum_{j=1}^{2J}\frac{1}{2}\left[\sigma_{1,j}^2-\sigma_{0,j}^2\right]\cdot\left[B_j\left(m\right)\right]^2$.

B.2 The discrete-time time-series processes for the ABE state variables and the ANS model coefficients

The data on both sides of equations 8 and 11 are Gaussian time-series processes under the ABE and ANS model assumptions. That is, for the steady-state components of the ABE model $\sum_{j=1}^{2J}\theta_j\left(t+\tau\right)$ $=\sum_{j=1}^{2J}\theta_j\left(t\right)+\sum_{j=1}^{2J}\int_t^{t+\tau}\sigma_{0,j}dz_{0,j}\left(v\right)dv$, where τ is an arbitrary increment of time, and for the Level coefficient of the ANS model $\beta_1\left(t+\tau\right)=\beta_1\left(t\right)+\int_t^{t+\tau}\sigma_1 dW_n\left(v\right)dv$. The stochastic integrals have closed forms; respectively $\sum_{j=1}^{2J}\int_t^{t+\tau}\sigma_{0,j}dz_{0,j}\left(v\right)dv=N\left(0,\tau\sum_{j=1}^{2J}\sigma_{0,j}^2\right)$, and $\int_t^{t+\tau}\sigma_1 dW_n\left(v\right)dv=N\left(0,\tau\sigma_1^2\right)$.

Regarding the ANS coefficients in equation 11, Krippner (2005) proves they are Gaussian time-series processes by deriving the underlying Gaussian vector autoregressive process for the ANS model coefficients over arbitrary increments of time τ. To prove that $\Delta X_{t+T_1,t+T_2}$ are also Gaussian time-series processes over arbitrary increments of time τ, begin with the intertemporal relationship for the expected path of the short rate from the HJM framework as derived in Krippner (2005), i.e:

$$E_{t+\tau}\left[r\left(t+\tau+m\right)\right]=E_t\left[r\left(t+\tau+m\right)\right]+\sum_{n=1}^{N}\int_t^{t+\tau}\sigma_n\left(v,m\right)dW_n\left(v\right) \tag{14}$$

This is intuitive: the expected path of the short rate would be realised but for the impact of unpredictable new information represented by the summation of stochastic integrals. These stochastic integrals do not have closed form solutions but $E_t\left[\int_t^{t+\tau}\sigma_n\left(v,m\right)dW_n\left(v\right)\right]=0$ (see Ross (1997) pp. 541-542), and each integral will be a summation of infinitesimal $\sigma_n\left(v,m\right)dW_n\left(v\right)$ increments expressible as $\varepsilon_n\left(t+\tau\right)\cdot\sigma_n\left(m\right)$, where $\varepsilon_n\left(t+\tau\right)$ is Gaussian by virtue of the Wiener processes being Gaussian.

The relationship in equation 14 will apply to each component j of the ABE model. For notational convenience, define $E_t\left[x_j\left(t+\tau+m\right)\right]=E_t\left[s_j\left(t+m\right)-\theta_j\left(t+m\right)\right]=E_t\left[-\theta_j\left(t+m\right)\right]+\theta_j\left(t\right)+\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)=\left[s_j\left(t\right)-\theta_j\left(t\right)\right]\cdot\exp\left(-\kappa_j m\right)$. Then applying the relationship from equation 14 for each component j in the ABE model gives:

---

[18] The ANS model could be extended arbitrarily by adding higher-order exponential-polynomial functions, which would be equivalent to adding terms in the Taylor expansion of the non-steady state components of the ABE model. In this sense, the approximation is natural, while the approximations based on other functions (e.g simple polynomials as in McCulloch (1971)), or Chebyshev polynomials as in Pham (1998)) would be "unnatural" because the addition of each higher-order term would not directly correspond to an extra term in the Taylor expansion.

$$E_{t+\tau}\left[x_j\left(t+\tau+m\right)\right]$$
$$= E_t\left[x_j\left(t+\tau+m\right)\right] + \int_t^{t+\tau} \exp\left(-\kappa_j\left[m-v\right]\right)\left[\sigma_{1,j}dz_{1,j}\left(v\right) - \sigma_{0,j}dz_{0,j}\left(v\right)\right] \quad (15)$$

where $E_t\left[\int_t^{t+\tau} \exp\left(-\kappa_j\left[m-v\right]\right)\left[\sigma_{1,j}\cdot z_{1,j}\left(v\right) - \sigma_{0,j}\cdot z_{0,j}\left(v\right)\right]\right] = 0$ and the integral will be expressible as $\left[\eta_{1,j}\left(t+\tau\right) - \eta_{0,j}\left(t+\tau\right)\right]\cdot\exp\left(-\kappa_j m\right) = \eta_j\left(t+\tau\right)\cdot\exp\left(-\kappa_j m\right)$. Hence:

$$x_j\left(t+\tau\right)\cdot\exp\left(-\kappa_j m\right) = x_j\left(t\right)\cdot\exp\left(-\kappa_j\left[\tau+m\right]\right) + \eta_j\left(t+\tau\right)\cdot\exp\left(-\kappa_j m\right) \quad (16\text{a})$$
$$x_j\left(t+\tau\right)\cdot\exp\left(-\kappa_j m\right) = \exp\left(-\kappa_j\tau\right)\cdot x_j\left(t\right)\cdot\exp\left(-\kappa_j m\right) + \eta_j\left(t+\tau\right)\cdot\exp\left(-\kappa_j m\right) \quad (16\text{b})$$
$$x_j\left(t+\tau\right) = \exp\left(-\kappa_j\tau\right)\cdot x_j\left(t\right) + \eta_j\left(t+\tau\right) \quad (16\text{c})$$

Therefore, $\eta_j\left(t+\tau\right)$ are Gaussian for arbitrary increments of time τ, which is the results required for the validity of using standard econometric estimation methods in section 3.3.

References

Aïssa, M. and Jouini, J. (2003), 'Structural breaks in the US inflation process', *Applied Economics Letters* **10**, 633–636.

Ang, A. and Piazzesi, M. (2003), 'A no-arbitrage vector autoregression of term structure dynamics with latent variables', *Journal of Monetary Economics* **50**, 745–787.

Bank for International Settlements (1999), *Zero-coupon Yield Curves: Technical Documentation*, Bank for International Settlements.

Berardi, A. and Esposito, M. (1999), 'A base model for multifactor specifications of the term structure', *Economic Notes* **28(2)**, 145–170.

Berk, J. (1998), 'The information content of the yield curve for monetary policy: a survey', *De Economist* **146**, 303–320.

Bernard, H. and Gerlach, S. (1998), 'Does the term structure predict recessions? The international evidence', *International Journal of Finance and Economics* **3**, 195–215.

Bordo, M. and Haubrich, J. (2004), 'The yield curve, recessions, and the credibility of the monetary regime: long run evidence 1875-1997', *NBER working paper series* **10431**.

Buraschi, A. and Jiltsov, A. (2005), 'Inflation risk premia and the expectations hypothesis', *Journal of Financial Economics* **75**, 429–490.

Chicago Mercentile Exchange (2005), 'An introduction to CME economic derivatives'.
URL: *http://www/cme.com*

Congressional Budget Office (2001), 'CBO's method for estimating potential output: an update'.
URL: *http://www/cbo.gov*

Cox, J., Ingersoll, J. and Ross, S. (1985*a*), 'An intertemporal general equilibrium model of asset prices', *Econometrica* **53**, 363–384.

Cox, J., Ingersoll, J. and Ross, S. (1985*b*), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–407.

Diebold, F. and Li, C. (2005), 'Forecasting the term structure of government bond yields', *Journal of Econometrics (forthcoming)* .

Diebold, F., Rudebusch, G. and Aruoba, S. (2005), 'The macroeconomy and the yield curve: a dynamic latent factor approach', *Journal of Econometrics (forthcoming)* .

Estrella, A. (2003), 'Why does the yield curve predict output and inflation?', *Working Paper, Federal Reserve Bank of New York* .

Estrella, A., Rodrigues, A. and Schich, S. (2003), 'How stable is the predictive power of the yield curve? Evidence from Germany and the United States', *Review of Economics and Statistics* **85(3)**, 629–644.

Fahmy, Y. and Kandi, M. (2003), 'The Fisher effect: new evidence and implications', *International Review of Economics and Finance* **12**, 451–465.

Fang, V. and Muljono, R. (2003), 'An empirical analysis of the Australian dollar swap spreads', *Pacific-Basin Finance Journal* **11**, 153–173.

Filipović, D. (2000), 'Exponential-polynomial families and the term structure of interest rates', *Bernoulli Journal* **6(6)**, 1081–1107.

Frankel, O. and O'Neill, J. (2002), 'Economic derivatives: what is their benefit?', *Derivatives Week* **11(38)**.

Goldman Sachs (2005), 'Economic derivatives: options on economic statistics'.
URL: *http://www/gs.com*

Goldstein, R. and Keirstead, W. (1997), 'On the term structure of interest rates in the presence of reflecting and absorbing boundaries', *Working paper, Ohio State University* .

Hamilton, J. and Kim, D. (2002), 'A re-examination of the predictability of economic activity using the yield spread', *Journal of Money, Credit, and Banking* **34(2)**, 340–360.

Heath, D., Jarrow, R. and Morton, A. (1992), 'Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation', *Econometrica* **60(1)**, 77–106.

Jardet, C. (2004), 'Why did the term structure of interest rates lose its predictive power?', *Economic Modelling* **21**, 509–524.

Krippner, L. (2005), 'A theoretically consistent version of the Nelson and Siegel class of yield curve models', *Applied Mathematical Finance (forthcoming)* .

Lai, K. (2004), 'On structural shifts and stationarity of the ex-ante real interest rate', *International Review of Economics and Finance* **13**, 217–228.

MacKinnon, J. (1996), 'Numerical distribution functions for unit root and cointegration tests', *Journal of Applied Econometrics* **11(6)**, 601–618.

McCulloch, J. (1971), 'Measuring the term structure of interest rates', *Journal of Business* **44**, 19–31.

Nakaota, H. (2005), 'The term structure of interest rates in Japan: the predictability of economic activity', *Japan and the World Economy* **17**, 311–326.

Nelson, C. and Siegel, A. (1987), 'Parsimonious modelling of yield curves', *Journal of Business* **October**, 473–489.

Paya, I., Matthews, K. and Peel, D. (2005), 'The term spread and real economic activity in the US inter-war periods', *Journal of Macroeconomics* **27(2)**, 331–343.

Pham, T. (1998), 'Estimation of the term structure of interest rates: an international perspective', *Journal of Multinational Financial Management* **8**, 265–283.

Rendu de Lint, C. and Stolin, D. (2003), 'The predictive power of the yield curve: a theoretical assessment', *Journal of Monetary Economics* **50**, 1603–1622.

Ross, S. (1997), *Introduction to Probability Models, Sixth Edition*, Academic Press.

Shiller, R. (1993), *Macro Markets: Creating Institutions for Managing Society's Largest Economic Risks*, Oxford University Press.

Shiller, R. (2003), *The New Financial Order: Risk in the 21st Century*, Princeton University Press.

Vasicek, O. (1977), 'An equilibrium characterisation of the term structure', *Journal of Financial Economics* **5**, 177–188.
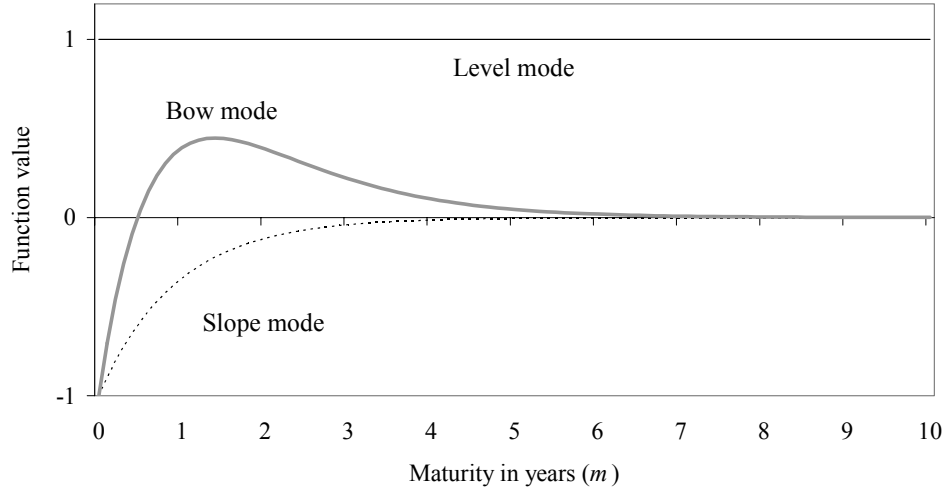
Figure 1: The Level, Slope, and Bow modes (i.e $g_1(\phi, m)$, $g_2(\phi, m)$, and $g_3(\phi, m)$ from the NS model) that are used to represent the expected path of the short rate in the ANS model. This illustration uses $\phi = 1.07$.
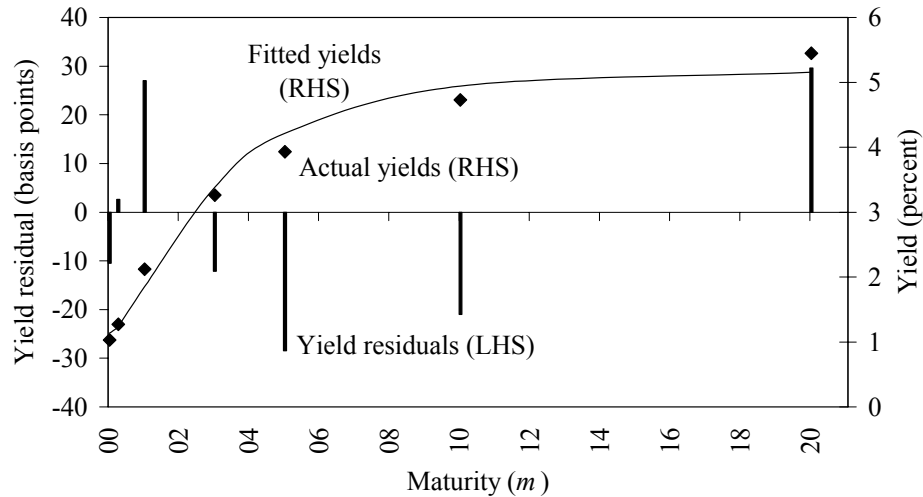
Figure 2: The yield curve data for the month of June 2004, and the "fitted" yields based on the estimated ANS model. The estimated Level, Slope, and Bow coefficients are, respectively, $\beta_1(\text{Jun-04}) = 5.87\%$, $\beta_2(\text{Jun-04}) = 6.80\%$, and $\beta_3(\text{Jun-04}) = -2.09\%$. The ANS parameters estimated over the entire sample are $\phi = 1.07$, $\rho_1 = 2.57\%$, $\sigma_1 = 0.79\%$, $\sigma_2 = 2.31\%$, and $\sigma_3 = 1.78\%$.
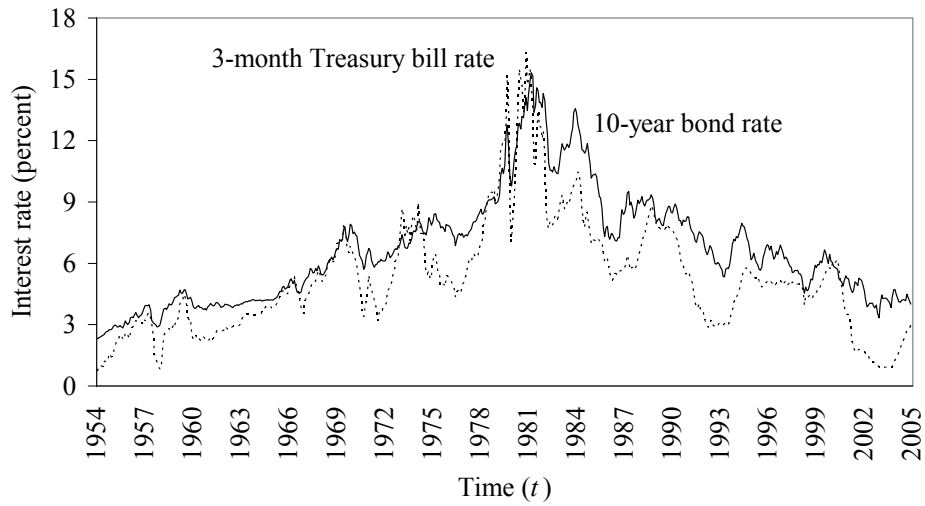
Figure 3: The time series of two of the seven interest rates that are used to estimate the time series of ANS Level, Slope, and Bow coefficients plotted in figure 4.
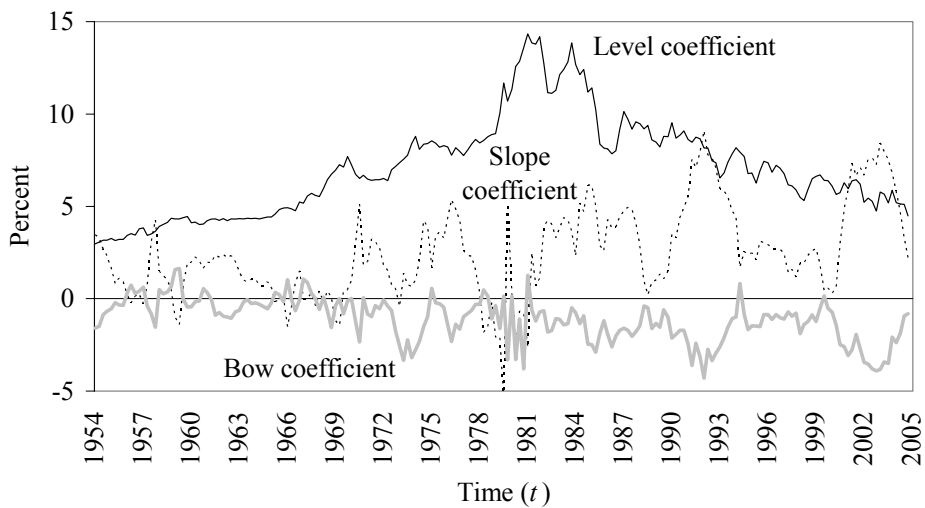
Figure 4: The time series of the estimated ANS Level, Slope, and Bow coefficients (i.e $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$). The ANS coefficients at each point in time are estimated using the seven points of yield curve data observed at that point in time.
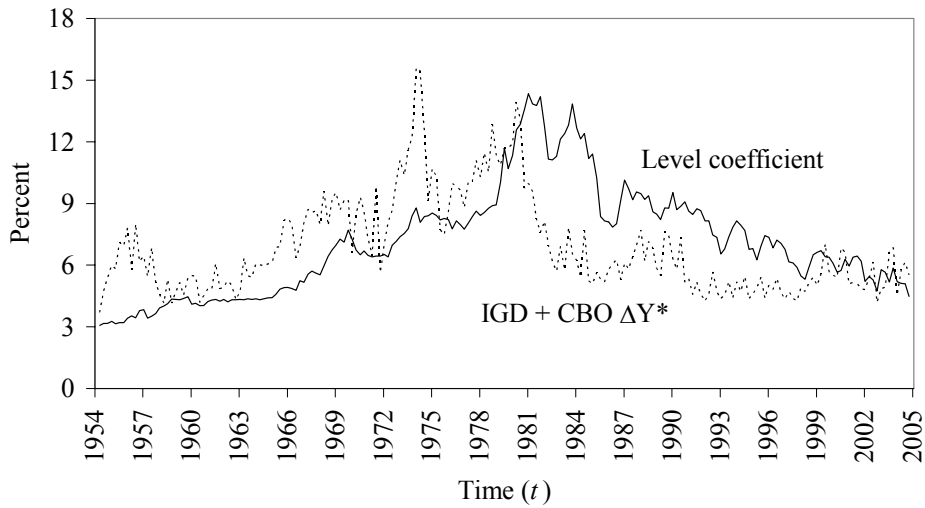
Figure 5: The time series of the ANS Level coefficient (i.e $\beta_1(t)$), and annualised quarterly inflation in the GDP deflator (IGD) plus annualised quarterly growth in Congressional Budget Office potential GDP (CBO ΔY^*). Note the apparent structural change between the two series from around the late-1970s/early-1980s, as discussed in section 4.1.
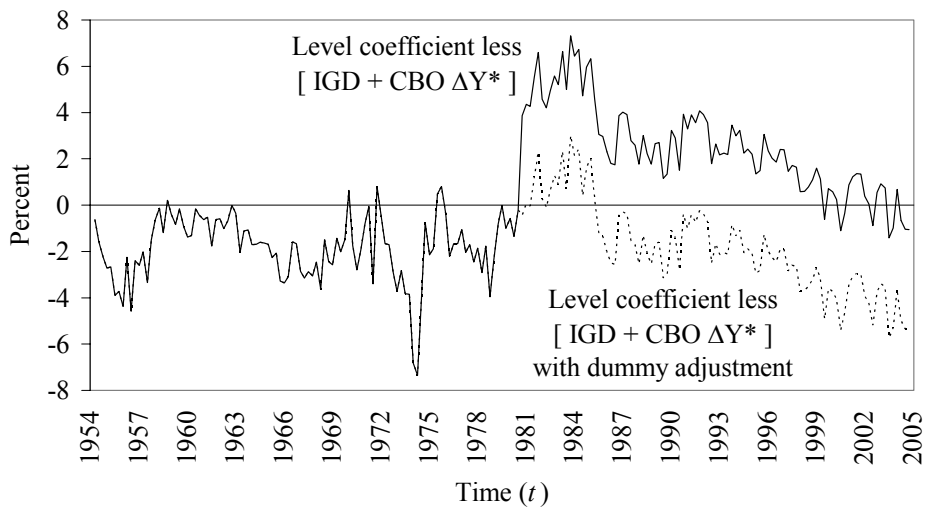
Figure 6: The time series of the ANS Level coefficient (i.e $\beta_1(t)$) less annualised quarterly inflation in the GDP deflator (IGD) plus annualised quarterly growth in Congressional Budget Office potential GDP (CBO ΔY^*). The alternative series allows for structural change in the difference between the two series from 1982:Q1, as discussed in section 4.1.
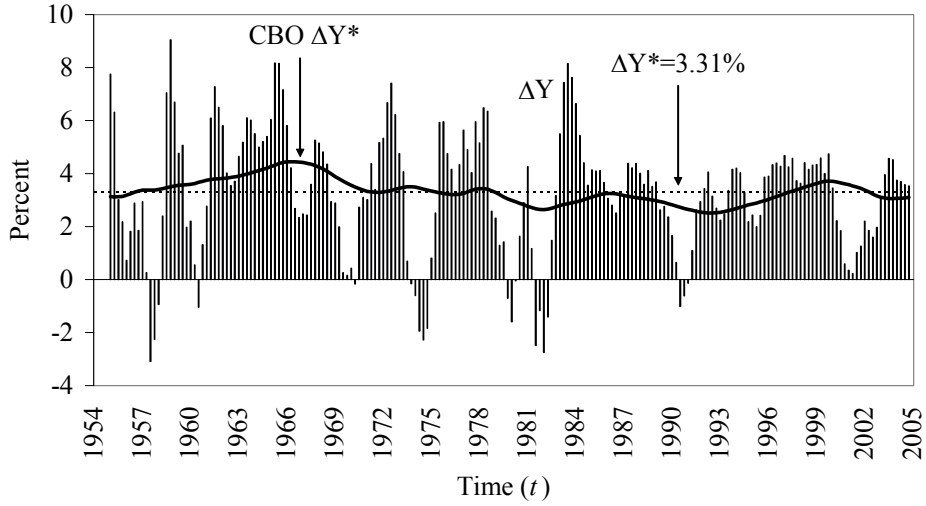
Figure 7: Output data used in the estimation of equation 11. ΔY is the annual growth in GDP, CBO ΔY^* is annual growth in Congressional Budget Office potential GDP, and $\Delta Y^* = 3.31\%$ is the estimate of constant potential growth. The difference between ΔY and the CBO ΔY^* is plotted in figure 8.
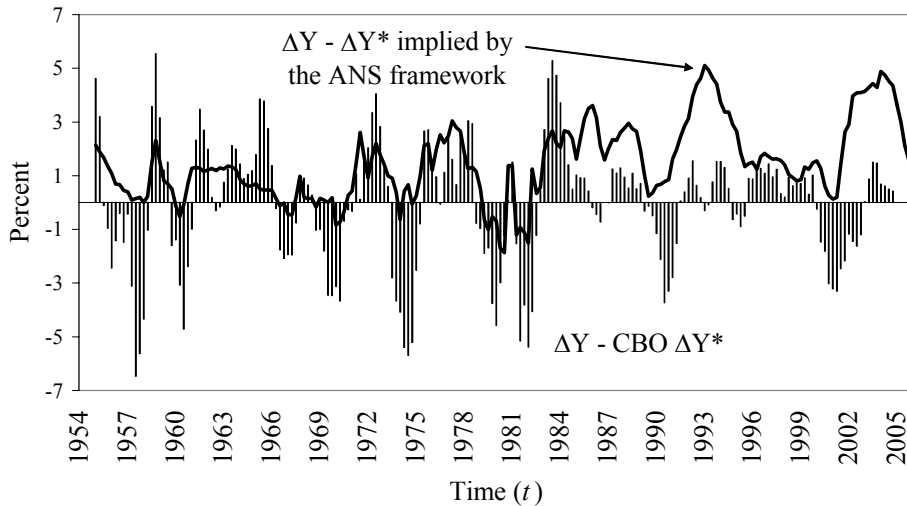
Figure 8: The time series of $\Delta Y_{t,t+1} - \Delta Y^*_{t,t+1} (= \Delta X_{t,t+1})$ based on the ΔY_t and CBO ΔY^*_t data plotted in figure 5, and $E_t \left[\Delta Y_{t,t+1} - \Delta Y^*_{t,t+1} \right]$ ($= E_t \left[\Delta X_{t,t+1} \right]$) as implied by the ANS framework. Note the apparent structural change in the relationship from around the late-1970s/early-1980s, as discussed in section 4.1.

Period	Full sample		Up to 1979:Q3		From 1984:Q1	
Unit root tests	ADF	PP	ADF	PP	ADF	PP
Level coefficient (β_1)	-1.7	-1.6	-0.2	-0.2	-2.4	-3.0 **
Pot. GDP growth (ΔY^*)	-2.8 *	-0.9	-1.9	-0.8	-2.7 *	-0.8
Dummy (D)	-1.0	-1.0	-	-	-	-
Δ GDP deflator (IGD)	-2.7 *	-3.2 **	-1.6	-2.2	-3.6 ***	-5.0 ***
Δ PCE deflator (PCE)	-2.8 *	-3.5 ***	-1.2	-1.9	-3.4 **	-4.7 ***
Δ PCEX deflator (PCEX)	-2.1	-2.6	-1.4	-1.7	-2.1	-2.9 **
IGD + ΔY^*	-2.7 *	-3.3 **	-1.8	-2.4	-3.9 ***	-5.3 ***
PCE + ΔY^*	-2.9 *	-3.6 ***	-1.5	-2.2	-3.6 ***	-5.0 ***
PCEX + ΔY^*	-2.2	-2.6 *	-1.7	-1.9	-2.3	-3.2 **
Cointegration tests	ADF	PP	ADF	PP	ADF	PP
β_1 - IGD	-2.7 *	-3.1 **	-3.6 ***	-4.5 ***	-2.8 *	-3.2 **
β_1 - PCE	-2.9 *	-3.5 **	-3.1 **	-4.4 ***	-3.2 **	-3.7 ***
β_1 - PCEX	-2.9 *	-3.2 **	-3.7 ***	-4.1 ***	-3.7 ***	-3.8 ***
β_1 - [IGD + ΔY^*]	-2.5	-2.8 *	-3.8 ***	-4.9 ***	-2.7 *	-3.0 **
β_1 - [PCE + ΔY^*]	-2.6 *	-3.2 **	-3.4 **	-4.7 ***	-3.1 **	-3.5 **
β_1 - [PCEX + ΔY^*]	-2.5	-2.8 *	-3.6 ***	-4.0 ***	-3.5 **	-3.6 ***
β_1 - IGD - D	-3.8 **	-4.7 ***	-	-	-	-
β_1 - PCE - D	-3.9 **	-5.1 ***	-	-	-	-
β_1 - PCEX - D	-4.0 ***	-4.9 ***	-	-	-	-
β_1 - [IGD + ΔY^*] - D	-3.6 **	-4.5 ***	-	-	-	-
β_1 - [PCE + ΔY^*] - D	-3.8 **	-4.9 ***	-	-	-	-
β_1 - [PCEX + ΔY^*] - D	-3.7 **	-4.5 ***	-	-	-	-

Table 1: Tests for cointegration between the ANS Level coefficient, and annualised quarterly measures of inflation with and without annualised quarterly growth in CBO potential GDP growth and with and without an estimated step dummy variable. ADF is augmented Dickey-Fuller, and PP is Phillips-Perron. ***, **, * respectively represent 1, 5, and 10 percent levels of significance.

Period	Full sample		Up to 1979:Q3		From 1984:Q1	
Unit root tests	ADF	PP	ADF	PP	ADF	PP
Level coefficient (β_1)	-1.7	-1.7	-0.3	-0.3	-0.8	-2.6 *
Pot. GDP growth (ΔY^*)	-2.0	-1.3	-1.0	-1.3	-1.9	-1.1
Dummy (D)	-0.9	-0.9	-	-	-	-
Δ GDP deflator (IGD)	-1.7	-1.9	-0.5	-1.0	-1.3	-2.1
Δ PCE deflator (PCE)	-2.0	-2.2	-0.6	-1.1	-1.3	-1.9
Δ PCEX deflator (PCEX)	-1.5	-1.7	-0.9	-1.1	-0.9	-1.3
IGD + ΔY^*	-1.7	-2.0	-0.8	-1.3	-1.5	-2.2
PCE + ΔY^*	-2.1	-2.3	-0.9	-1.4	-1.5	-2.0
PCEX + ΔY^*	-1.5	-1.8	-1.3	-1.4	-1.1	-1.5
Cointegration tests	ADF	PP	ADF	PP	ADF	PP
β_1 - IGD	-2.1	-2.1	-2.7 *	-2.5	-1.0	-2.4
β_1 - PCE	-2.4	-2.4	-2.8 *	-2.7 *	-0.9	-3.0 **
β_1 - PCEX	-2.2	-2.5	-2.8 *	-2.9 *	-2.0	-3.7 ***
β_1 - [IGD + ΔY^*]	-2.0	-1.9	-3.1 **	-2.9 *	-0.9	-2.3
β_1 - [PCE + ΔY^*]	-2.3	-2.2	-1.4	-1.8	-1.4	-2.3
β_1 - [PCEX + ΔY^*]	-2.2	-2.2	-1.4	-1.7	-1.5	-2.7 *
β_1 - IGD - D	-2.7	-2.7	-	-	-	-
β_1 - PCE - D	-3.1 *	-3.1 *	-	-	-	-
β_1 - PCEX - D	-3.3 *	-3.4 **	-	-	-	-
β_1 - [IGD + ΔY^*] - D	-2.6	-2.7	-	-	-	-
β_1 - [PCE + ΔY^*] - D	-3.1 *	-3.0	-	-	-	-
β_1 - [PCEX + ΔY^*] - D	-3.3 *	-3.0	-	-	-	-

Table 2: Tests for cointegration between the ANS Level coefficient, and annual measures of inflation with and without annual growth in potential GDP growth and with and without an estimated step dummy variable. ADF is augmented Dickey-Fuller, and PP is Phillips-Perron. ***, **, * respectively represent 1, 5, and 10 percent levels of significance.

Horizon	$q_n(T_1,T_2)$		R^2	Constant	ANS	Dummy	ANS coeff.
T_1 , T_2	Slope	Bow	in %	coefficient	coefficient	coefficient	less 1
0 - 0.25	0.88	0.65	10.0	-0.64 *	0.76 ***	-1.16 **	-0.24
0.25 - 0.5	0.67	0.14	15.7	-0.95 **	1.10 ***	-1.81 ***	0.10
0.5 - 0.75	0.51	-0.17	8.5	-0.83 *	0.89 ***	-1.34 **	-0.11
0.75 - 1	0.39	-0.34	6.3	-0.78	0.85 ***	-1.13	-0.15
1 - 1.25	0.30	-0.42	5.7	-0.76	0.91 ***	-1.05	-0.09
1.25 - 1.5	0.23	-0.44	1.1	-0.42	0.46	-0.39	-0.54
1.5 - 1.75	0.18	-0.43	1.0	-0.34	0.50	-0.45	-0.50
1.75 - 2	0.13	-0.40	1.3	-0.38	0.67 *	-0.51	-0.33
2 - 2.25	0.10	-0.36	0.5	-0.23	0.49	-0.34	-0.51
2.25 - 2.5	0.08	-0.32	0.0	-0.04	-0.18	0.14	-1.18 *
2.5 - 2.75	0.06	-0.28	0.0	-0.06	-0.11	0.07	-1.11 *
2.75 - 3	0.05	-0.24	0.8	-0.29	1.13	-0.38	0.13
0 - 1	0.61	0.07	25.4	-0.82 **	0.93 ***	-1.40 ***	-0.07
1 - 2	0.21	-0.43	4.9	-0.48	0.66 ***	-0.59	-0.34
2 - 3	0.07	-0.30	0.2	-0.15	0.28	-0.12	-0.72

Table 3: Full-sample estimates of equation 11 using a step dummy variable and $\Delta X_{t+T_1,t+T_2}$ based on CBO potential output growth. ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.

Horizon	Up to 1979:Q3				From 1984:Q1			
T_1, T_2	R^2 in %	Const. coeff.	ANS coeff.	ANS cf. less 1	R^2 in %	Const. coeff.	ANS coeff.	ANS cf. less 1
0 - 0.25	17.8	-1.31 ***	1.53 ***	0.53	4.2	-0.61	0.27 *	-0.73 ***
0.25 - 0.5	17.2	-1.57 ***	1.82 ***	0.82 **	5.1	-0.77	0.31 *	-0.69 ***
0.5 - 0.75	9.2	-1.28 **	1.47 ***	0.47	8.0	-1.01	0.41 **	-0.59 ***
0.75 - 1	13.1	-1.55 ***	1.93 ***	0.93 *	6.2	-0.91	0.40 *	-0.60 **
1 - 1.25	4.7	-0.96	1.26 **	0.26	5.9	-0.90	0.44 *	-0.56 **
1.25 - 1.5	0.8	-0.45	0.58	-0.42	4.3	-0.78	0.44	-0.56 *
1.5 - 1.75	0.2	-0.22	0.31	-0.69	9.4	-1.16	0.75 **	-0.25
1.75 - 2	0.3	-0.26	0.47	-0.53	7.4	-1.05	0.81 *	-0.19
2 - 2.25	0.0	-0.09	0.17	-0.83	6.7	-1.02	0.94 **	-0.06
2.25 - 2.5	1.9	0.36	-1.58	-2.58	1.3	-0.48	0.51	-0.49
2.5 - 2.75	0.7	0.19	-1.16	-2.16 *	0.2	-0.24	0.27	-0.73
2.75 - 3	0.3	-0.16	0.91	-0.09	1.1	-0.43	0.70	-0.30
0 - 1	38.0	-1.56 ***	1.80 ***	0.80 **	12.1	-0.80	0.33 *	-0.67 ***
1 - 2	2.7	-0.43	0.68 *	-0.32	15.2	-1.00	0.61 **	-0.39
2 - 3	0.2	0.07	-0.37	-1.37	3.8	-0.56	0.66	-0.34

Table 4: Sub-sample stimates of equation 11 using $\Delta X_{t+T_1, t+T_2}$ based on CBO potential output growth. ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.

Horizon	$q_n(T_1,T_2)$		R^2	Constant	ANS	Dummy	ANS coeff.
T_1, T_2	Slope	Bow	in %	coefficient	coefficient	coefficient	less 1
0 - 0.25	0.88	0.65	8.7	-0.41	0.71 ***	-1.49 ***	-0.29 *
0.25 - 0.5	0.67	0.14	13.5	-0.70	1.03 ***	-2.09 ***	0.03
0.5 - 0.75	0.51	-0.17	6.8	-0.57	0.80 ***	-1.59 **	-0.20
0.75 - 1	0.39	-0.34	4.8	-0.51	0.75 ***	-1.37 *	-0.25
1 - 1.25	0.30	-0.42	4.3	-0.50	0.79 ***	-1.29 *	-0.21
1.25 - 1.5	0.23	-0.44	0.6	-0.16	0.32	-0.63	-0.68 **
1.5 - 1.75	0.18	-0.43	0.7	-0.07	0.34	-0.70	-0.66 **
1.75 - 2	0.13	-0.40	0.9	-0.11	0.48	-0.76	-0.52
2 - 2.25	0.10	-0.36	0.5	0.04	0.27	-0.60	-0.73 *
2.25 - 2.5	0.08	-0.32	0.5	0.22	-0.43	-0.13	-1.43 **
2.5 - 2.75	0.06	-0.28	0.4	0.20	-0.40	-0.21	-1.40 **
2.75 - 3	0.05	-0.24	0.7	-0.04	0.79	-0.68	-0.21
0 - 1	0.61	0.07	20.5	-0.57	0.86 ***	-1.68 ***	-0.14
1 - 2	0.21	-0.43	3.1	-0.21	0.50 **	-0.84	-0.50 **
2 - 3	0.07	-0.30	0.7	0.11	0.00	-0.40	-1.00 *

Table 5: Full-sample estimates of equation 11 using a step dummy variable and $\Delta X_{t+T_1, t+T_2}$ based on constant potential output growth (i.e $\Delta Y_t^* = 3.31\%$). ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.

	Up to 1979:Q3				From 1984:Q1			
Horizon	R^2	Const.	ANS	ANS cf.	R^2	Const.	ANS	ANS cf.
T_1, T_2	in %	coeff.	coeff.	less 1	in %	coeff.	coeff.	less 1
0 - 0.25	15.3	-0.90 *	1.43 ***	0.43	1.7	-0.60	0.17	-0.83 ***
0.25 - 0.5	14.3	-1.13 *	1.67 ***	0.67 *	2.3	-0.71	0.21	-0.79 ***
0.5 - 0.75	7.0	-0.81	1.30 ***	0.30	4.1	-0.91	0.30	-0.70 ***
0.75 - 1	10.3	-1.08 *	1.74 ***	0.74	2.7	-0.77	0.27	-0.73 ***
1 - 1.25	3.2	-0.49	1.05	0.05	2.8	-0.79	0.31	-0.69 ***
1.25 - 1.5	0.3	0.02	0.35	-0.65	1.9	-0.68	0.30	-0.70 **
1.5 - 1.75	0.0	0.24	0.07	-0.93	4.5	-0.96	0.54	-0.46
1.75 - 2	0.1	0.20	0.20	-0.80	2.6	-0.77	0.49	-0.51
2 - 2.25	0.0	0.37	-0.13	-1.13	2.4	-0.74	0.57	-0.43
2.25 - 2.5	2.7	0.80	-1.90	-2.90 *	0.1	-0.29	0.13	-0.87
2.5 - 2.75	1.2	0.63	-1.51	-2.51 **	0.5	0.07	-0.39	-1.39 **
2.75 - 3	0.1	0.26	0.55	-0.45	0.2	-0.34	0.30	-0.70
0 - 1	29.3	-1.07 *	1.62 ***	0.62 *	5.7	-0.77	0.23	-0.77 ***
1 - 2	1.1	0.04	0.45	-0.55	7.6	-0.90	0.44	-0.56 *
2 - 3	0.7	0.51	-0.67	-1.67	0.8	-0.45	0.31	-0.69

Table 6: Sub-sample estimates of equation 11 using $\Delta X_{t+T_1, t+T_2}$ based on constant potential output growth (i.e $\Delta Y_t^* = 3.31\%$). ***, **, * respectively represent 1, 5, and 10 percent two-tailed levels of significance.